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Fuzzy nonlinear set-valued variational inclusions[☆]

Byung Soo Lee [a,*], M. Firdosh Khan [b], Salahuddin [c]

[a] Department of Mathematics, Kyungsung University, Busan 608-736, Republic of Korea
[b] S.S. School (Boys) Aligarh Muslim University, Aligarh-202002, India
[c] Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

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ABSTRACT

The purpose of this paper is to study a new class of fuzzy nonlinear set-valued variational inclusions in real Banach spaces. By using the fuzzy resolvent operator techniques for m-accretive mappings, we establish the equivalence between fuzzy nonlinear set-valued variational inclusions and fuzzy resolvent operator equation problem. Applying this equivalence and Nadler's theorem, we suggest some iterative algorithms for solving fuzzy nonlinear set-valued variational inclusions in real Banach spaces. By using the inequality of Petryshyn, the existence of solutions for these kinds of fuzzy nonlinear set-valued variational inclusions without compactness is proved and convergence criteria of iterative sequences generated by the algorithm are also discussed.

1. Introduction and preliminaries

Recently, there have been many research on the variational inequality problems and complementarity problems for fuzzy mappings [1–13]. In particular, Chang [6] introduced fuzzy quasivariational inclusions and established existence problem and iterative approximation problem of solutions for such inclusions by using the resolvent operator technique, Nadler's fixed point theorem and analytic techniques in Banach spaces. In 2004, Liu et al. [12] introduced fuzzy mixed quasivariational inclusions and some mixed resolvent equations and established some existence theorems of solutions for the inclusions in Hilbert spaces.

In this paper, we introduce a class of fuzzy nonlinear set-valued variational inclusions in real Banach spaces. By using the fuzzy resolvent operator techniques for m-accretive mappings, we establish the equivalence between fuzzy nonlinear set-valued variational inclusions and fuzzy resolvent operator equation problem. Applying this equivalence and Nadler's theorem, we suggest some iterative algorithms for solving fuzzy nonlinear set-valued variational inclusions in real Banach spaces. By using the inequality of Petryshyn [14,15], the existence of solutions for the inclusions without compactness is proved and convergence criteria of iterative sequences generated by the algorithm are also discussed.

A mapping T from a set X to the collection $\mathcal{F}(X) = \{B : X \to [0, 1] \text{ a function}\}$ of fuzzy sets over X is called a fuzzy mapping, which means that for each $x \in X$ a fuzzy set $T(x)$, denoted by T_x, is a function from X to $[0, 1]$. For each $y \in X$, $T_x(y)$ denotes the membership grade of y in T_x.

A fuzzy mapping $T : X \to \mathcal{F}(X)$ is said to be closed if for each $x \in X$, the function $y \to T_x(y)$ is upper semicontinuous, that is, for any given net $\{y_\alpha\} \subset X$, satisfying $y_\alpha \to y_0 \in X$, we have

$$\limsup_{\alpha} T_x(y_\alpha) \leq T_x(y_0).$$

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* Corresponding author. Tel.: +82 51 66 4613; fax: +82 51 622 7510.
E-mail addresses: bslee@ks.ac.kr (B.S. Lee), khan_mfk@yahoo.com (M.F. Khan), salahuddin12@mailcity.com (Salahuddin).

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For $B \in \mathcal{F}(X)$ and $\lambda \in [0, 1]$, the set $(B)_\lambda = \{x \in X : B(x) \geq \lambda\}$ is called a λ-cut set of B.

Let $T : X \to \mathcal{F}(X)$ be a closed fuzzy mapping satisfying the following condition:

Condition $(*)$. If there exists a function $a : X \to [0, 1]$ such that for each $x \in X$, the set $(T_x)_{a(x)} = \{y \in X : T_x(y) \geq a(x)\}$ is a nonempty bounded subset of X.

Remark 1.1. Let X be a normed vector space. If T is a closed fuzzy mapping satisfying the condition $(*)$, then for each $x \in X$, the set $(T_x)_{a(x)}$ belongs to the collection $CB(X)$ of all nonempty closed and bounded subsets of X. In fact, let $\{y_\alpha\} \subset (T_x)_{a(x)}$ be a net and $y_\alpha \to y_0 \in X$, then $(T_x)(y_\alpha) \geq a(x)$ for each α. Since T is closed, we have

$$T_x(y_0) \geq \limsup_{\alpha} T_x(y_\alpha) \geq a(x),$$

which implies that $y_0 \in (T_x)_{a(x)}$ and so $(T_x)_{a(x)} \in CB(X)$.

Let $J : X \to 2^{X^*}$ be the normalized duality mapping defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|\,\|f\|,\ \|f\| = \|x\|\} \quad \text{for } x \in X,$$

where $\langle \cdot, \cdot\rangle$ is the dual pairing between X and its dual X^*.

If X^* is uniformly convex, then J is single-valued and uniformly continuous on a bounded subset of X. We denote a single-valued normalized duality mapping by j.

Let X be a real Banach space. Let T, F, E and $P : X \to \mathcal{F}(X)$ be closed fuzzy mappings satisfying the condition $(*)$ with functions a, b, c and $e : X \to [0, 1]$, respectively, and $G, g : X \to X$ be surjective single-valued mappings. Let $A : X \times X \to 2^X$ be an m-accretive mapping with respect to the first argument. For a given nonlinear mapping $N : X \times X \to X$, we consider a problem of finding $x, u, v, w, q \in X$ such that

$$T_x(u) \geq a(x), \qquad F_x(v) \geq b(x), \qquad E_x(w) \geq c(x), \qquad P_x(q) \geq e(x),$$

i.e., $u \in (T_x)_{a(x)}$, $v \in (F_x)_{b(x)}$, $w \in (E_x)_{c(x)}$, $q \in (P_x)_{e(x)}$,

$$0 \in G(w) - N(u, v) + A(g(x), q). \tag{1.1}$$

Problem (1.1) is called a fuzzy nonlinear variational inclusion.

Problem (1.1) includes many types of variational inclusions, variational inequalities and complementarity problems for fuzzy mappings [1,5–7,12].

Putting $a(x) = b(x) = c(x) = e(x) = 1$ for all $x \in X$, problem (1.1) includes many kinds of variational inclusions, variational inequalities and complementarity problems [16–21].

Definition 1.1. Let $A : D(A) \subset X \to 2^X$ be a set-valued mapping, where $D(A)$ is the domain of A.

(i) A is said to be accretive if for any $x, y \in D(A)$, $u \in A(x)$, $v \in A(y)$ there exists a $j(x - y) \in J(x - y)$ such that

$$\langle u - v,\ j(x - y)\rangle \geq 0.$$

(ii) A is said to be m-accretive if A is accretive and $(I + \rho A)(D(A)) = X$ for every (equivalently, for some) $\rho > 0$, where I is the identity mapping.

Definition 1.2 (*[22]*)**.** Let $A : D(A) \subset X \to 2^X$ be an m-accretive mapping. For any given $\eta > 0$, the mapping $J_A : X \to D(A)$ associated with A defined by

$$J_A(x) = (I + \eta A)^{-1}(x) \quad \text{for } x \in X,$$

is called the resolvent operator of A.

Remark 1.2. Barbu [22] pointed out that if A is an m-accretive mapping, then for every $\eta > 0$ the operator $(I + \eta A)^{-1}$ is well-defined, single-valued and nonexpansive on the range $R(I + \eta A)$, i.e.,

$$\|J_A(x) - J_A(y)\| \leq \|x - y\| \quad \text{for } x, y \in R(I + \eta A).$$

From Remark 1.2, we have the following result.

Proposition 1.1. *Let $A : D(A) \subset X \times X \to 2^X$ be an m-accretive mapping with respect to the first argument. For a constant $\eta > 0$ we get*

$$J_{A(\cdot,z)} = (I + \eta A(\cdot, z))^{-1} \quad \text{for } z \in X.$$

Then for any given $z \in X$, the resolvent operator $J_{A(\cdot,z)} : X \to D(A)(\subset X \times X)$ is well-defined, single-valued and nonexpansive, that is,

$$\|J_{A(\cdot,z)}(x) - J_{A(\cdot,z)}(y)\| \leq \|x - y\| \quad \text{for } x, y \in X.$$

Definition 1.3. Let $T : X \to \mathcal{F}(X)$ be a closed fuzzy mapping satisfying the condition $(*)$ with a function $a : X \to [0, 1]$. T is said to be ξ-Lipschitz continuous if for any $x, y \in X$

$$H((T_x)_{a(x)}, (T_y)_{a(y)}) \leq \xi \|x - y\|,$$

where $\xi > 0$ is a constant and H is the Hausdorff metric on $CB(X)$.

Definition 1.4. Let X be a real Banach space and $g : X \to X$ a single-valued mapping;

(i) g is said to be accretive if

$$\langle g(x) - g(y), j(x - y)\rangle \geq 0$$

for $j(x - y) \in J(x - y)$.

(ii) g is said to be γ-strongly accretive if there exists a constant $\gamma > 0$ such that

$$\langle g(x) - g(y), j(x - y)\rangle \geq \gamma \|x - y\|^2 \quad \text{for } x, y \in X,\ j(x - y) \in J(x - y).$$

Related to the fuzzy nonlinear variational inclusion problem (1.1) we consider the following fuzzy resolvent operator equation problem:

Find $x, z, u, v, w, q \in X$ such that

$$\begin{aligned} &(T_x)(u) \geq a(x), \qquad (F_x)(v) \geq b(x), \qquad (E_x)(w) \geq c(x), \qquad (P_x)(q) \geq e(x), \\ &G(w) + \eta^{-1} R_{A(\cdot,q)}(z) = N(u, v), \end{aligned} \tag{1.2}$$

where $\eta > 0$ is a constant and $R_{A(\cdot,q)} = (I - J_{A(\cdot,q)})$.

The following known lemma plays an important role in proving our main results.

Lemma 1.1 ([14,15]). *Let X be a real Banach space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then for any $x, y \in X$*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle \quad \text{for } j(x + y) \in J(x + y).$$

2. Main results

Now we establish the equivalence between fuzzy nonlinear set-valued variational inclusions and fuzzy resolvent operator equation problem.

Theorem 2.1. *Let X be a Banach space. Assume that $T, F, E, P : X \to \mathcal{F}(X)$ are fuzzy mappings satisfying the condition $(*)$, with functions a, bc and $e : X \to [0, 1]$, respectively. Let $G, g : X \to X$ and $N : X \times X \to X$ be single-valued mappings. Then the followings are equivalent:*

(i) *(x, u, v, w, q), where $x \in X$, $(T_x)(u) \geq a(x)$, $(F_x)(v) \geq b(x)$, $(E_x)(w) \geq c(x)$, $(P_x)(q) \geq e(x)$ is a solution of problem (1.1),*

(ii) *$x \in X$ is a fixed point of a mapping $S : X \to 2^X$ defined by*

$$S(x) = G(w) - N(u, v) + A(g(x), q) + x, \tag{2.1}$$

(iii) *(x, u, v, w, q), where $x \in X$, $(T_x)(u) \geq a(x)$, $(F_x)(v) \geq b(x)$, $(E_x)(w) \geq c(x)$, $(P_x)(q) \geq e(x)$, is a solution of the following equation*

$$g(x) = J_{A(\cdot,q)}[g(x) - \eta(G(w) - N(u, v))], \tag{2.2}$$

(iv) *(x, z, u, v, w, q), where $x, z \in X$, $(T_x)(u) \geq a(x)$, $(F_x)(v) \geq b(x)$, $(E_x)(w) \geq c(x)$, $(P_x)(q) \geq e(x)$, is a solution of problem (1.2), where*

$$\begin{aligned} &z = g(x) - \eta(G(w) - N(u, v)), \\ &g(x) = J_{A(\cdot,q)}(z). \end{aligned} \tag{2.3}$$

Proof. (i) $\Rightarrow$ (ii) Adding x to both sides of (1.1), we have

$$\begin{aligned} &0 \in G(w) - N(u, v) + A(g(x), q) \\ &\Rightarrow x \in G(w) - N(u, v) + A(g(x), q) + x = S(x). \end{aligned}$$

(ii) $\Rightarrow$ (iii) Let x be a fixed point of S, then

$$\begin{aligned} &x \in G(w) - N(u, v) + A(g(x), q) + x \Rightarrow 0 \in G(w) - N(u, v) + A(g(x), q) \\ &\Rightarrow 0 \in \eta(G(w) - N(u, v)) + \eta A(g(x), q) \\ &\Rightarrow 0 \in -(g(x) - \eta(G(w) - N(u, v))) + g(x) + \eta A(g(x), q) \\ &\Rightarrow 0 \in -(g(x) - \eta(G(w) - N(u, v))) + (I + \eta A(\cdot, q))(g(x)). \end{aligned}$$

Hence $g(x) = J_{A(\cdot,q)}[g(x) - \eta(G(w) - N(u, v))]$.

(iii) $\Rightarrow$ (iv) Taking $z = g(x) - \eta(G(w) - N(u, v))$, from (2.2) we have $g(x) = J_{A(\cdot,q)}(z)$ so,

$$z = J_{A(\cdot,q)}(z) - \eta(G(w) - N(u, v)),$$

which implies that

$$(I - J_{A(\cdot,q)})(z) = -\eta(G(w) - N(u, v))$$
$$\eta^{-1}(I - J_{A(\cdot,q)})(z) = -G(w) + N(u, v)$$
$$\Rightarrow G(w) + \eta^{-1}(I - J_{A(\cdot,q)})(z) = N(u, v)$$
$$\Rightarrow G(w) + \eta^{-1}R_{A(\cdot,q)}(z) = N(u, v).$$

Consequently, (x, z, u, v, w, q) is a solution of the fuzzy resolvent operator equation problem (1.2).
(iv) $\Rightarrow$ (i), From (2.3) we have

$$\begin{aligned} g(x) &= J_{A(\cdot,q)}(z) \\ &= J_{A(\cdot,q)}[g(x) - \eta(G(w) - N(u, v))], \end{aligned}$$

i.e.,

$$g(x) = (I + \eta A(\cdot, q))^{-1}[g(x) - \eta(G(w) - N(u, v))]$$

so

$$[g(x) - \eta(G(w) - N(u, v))] \in [I + \eta A(\cdot, q)](g(x)),$$

which implies

$$0 \in G(w) - N(u, v) + A(g(x), q).$$

Therefore (x, u, v, w, q), where $x \in X$, $(T_x)(u) \geq a(x)$, $(F_x)(v) \geq b(x)$, $(E_x)(w) \geq c(x)$, $(P_x)(q) \geq e(x)$, is a solution of (1.1).

We now invoke Lemma 1.1 and (2.3) to suggest the following algorithm for solving problem (1.1) in Banach spaces. □

Algorithm 2.1. We assume that g is surjective. For any given $x_0, z_0 \in X$, $u_0 \in (T_{x_0})_{a(x_0)}$, $v_0 \in (F_{x_0})_{b(x_0)}$, $w_0 \in (E_{x_0})_{c(x_0)}$, $q_0 \in (P_{x_0})_{e(x_0)}$, let

$$z_1 = g(x_0) - \eta(G(w_0) - N(u_0, v_0)).$$

Since g is surjective, there exists $x_1 \in X$ such that

$$g(x_1) = J_{A(\cdot,q_0)}(z_1).$$

On the other hand, by Nadler [23], there exist $u_1 \in (T_{x_1})_{a(x_1)}$, $v_1 \in (F_{x_1})_{b(x_1)}$, $w_1 \in (E_{x_1})_{c(x_1)}$ and $q_1 \in (P_{x_1})_{e(x_1)}$ such that

$$\|u_0 - u_1\| \leq (1 + 1)H\Big((T_{x_0})_{a(x_0)}, (T_{x_1})_{a(x_1)}\Big),$$
$$\|v_0 - v_1\| \leq (1 + 1)H\Big((F_{x_0})_{b(x_0)}, (F_{x_1})_{b(x_1)}\Big),$$
$$\|w_0 - w_1\| \leq (1 + 1)H\Big((E_{x_0})_{c(x_0)}, (E_{x_1})_{c(x_1)}\Big),$$
$$\|q_0 - q_1\| \leq (1 + 1)H\Big((P_{x_0})_{e(x_0)}, (P_{x_1})_{e(x_1)}\Big).$$

Let

$$z_2 = g(x_1) - \eta(G(w_1) - N(u_1, v_1)).$$

Again by the surjectivity of g, there exists $x_2 \in X$ such that

$$g(x_2) = J_{A(\cdot,q_1)}(z_2).$$

Similarly, there exist $u_2 \in (T_{x_2})_{a(x_2)}$, $v_2 \in (F_{x_2})_{b(x_2)}$, $w_2 \in (E_{x_2})_{c(x_2)}$ and $q_2 \in (P_{x_2})_{e(x_2)}$ such that

$$\|u_1 - u_2\| \leq \left(1 + \frac{1}{2}\right)H\Big((T_{x_1})_{a(x_1)}, (T_{x_2})_{a(x_2)}\Big),$$
$$\|v_1 - v_2\| \leq \left(1 + \frac{1}{2}\right)H\Big((F_{x_1})_{b(x_1)}, (F_{x_2})_{b(x_2)}\Big),$$
$$\|w_1 - w_2\| \leq \left(1 + \frac{1}{2}\right)H\Big((E_{x_1})_{c(x_1)}, (E_{x_2})_{c(x_2)}\Big),$$
$$\|q_1 - q_2\| \leq \left(1 + \frac{1}{2}\right)H\Big((P_{x_1})_{e(x_1)}, (P_{x_2})_{e(x_2)}\Big).$$

Continuing this process, we obtain sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{q_n\}$ and $\{z_n\}$ in X such that for all $n \geq 0$,

(i) $u_n \in (T_{x_n})_{a(x_n)}$

$$\|u_n - u_{n-1}\| \le \left(1+\frac{1}{n}\right) H\Big((T_{x_n})_{a(x_n)}, (T_{x_{n-1}})_{a(x_{n-1})}\Big),$$

(ii) $v_n \in (F_{x_n})_{b(x_n)}$

$$\|v_n - v_{n-1}\| \le \left(1+\frac{1}{n}\right) H\Big((F_{x_n})_{b(x_n)}, (F_{x_{n-1}})_{b(x_{n-1})}\Big),$$

(iii) $w_n \in (E_{x_n})_{c(x_n)}$

$$\|w_n - w_{n-1}\| \le \left(1+\frac{1}{n}\right) H\Big((E_{x_n})_{c(x_n)}, (E_{x_{n-1}})_{c(x_{n-1})}\Big),$$

(iv) $q_n \in (P_{x_n})_{e(x_n)}$

$$\|q_n - q_{n-1}\| \le \left(1+\frac{1}{n}\right) H\Big((P_{x_n})_{e(x_n)}, (P_{x_{n-1}})_{e(x_{n-1})}\Big),$$

(v)

$$z_{n+1} = g(x_n) - \eta(G(w_n) - N(u_n, v_n)), \tag{2.4}$$

(vi)

$$g(x_{n+1}) = J_{A(\cdot,q_n)}(z_{n+1}). \tag{2.5}$$

Now we consider our main result.

Theorem 2.2. *Let X be a real Banach space. Let T, F, E and $P : X \to \mathcal{F}(X)$ be closed fuzzy mappings satisfying the condition $(*)$ with functions a, b, c and $e : X \to [0, 1]$, respectively. Let $N : X \times X \to X$ be continuous, $G, g : X \to X$ surjective and $A : X \times X \to 2^X$ m-accretive with respect to the first argument.*

Suppose that the following conditions hold;

(i) *g is β-Lipschitz continuous and γ-strongly accretive, $0 < \beta < 1$, $\gamma \in (0, 1) \setminus \{\frac{1}{2}\}$,*
(ii) *T, F, E and P are Lipschitz continuous with their Lipschitz constants ξ, α, ζ and σ, respectively,*
(iii) *G is ϵ-Lipschitz continuous,*
(iv) *for any given $y \in X$, a mapping $x \to N(x, y)$ is λ-Lipschitz continuous,*
(v) *for any given $x \in X$, a mapping $y \to N(x, y)$ is μ-Lipschitz continuous.*

If the following conditions

$$\text{(a) } \|J_{A(\cdot,x)}(z) - J_{A(\cdot,y)}(z)\| \le \kappa\|x - y\|, \quad \text{for } x, y, z \in X, \kappa > 0, \tag{2.6}$$

$$\text{(b) } \left[\frac{1}{-1+2\gamma}\left(\frac{c+\sqrt{c^2+4\beta^2}}{2} + 2\kappa\sigma^2\right)\right]^{\frac{1}{2}} < 1 \tag{2.7}$$

for $t = \epsilon^2\zeta^2 + \lambda^2\xi^2 + \mu^2\alpha^2 + 2\lambda\xi\mu\alpha > 0$ and $c = 2\sqrt{2}\eta t$ are satisfied, then there exist $x, z \in X$, $u \in (T_x)_{a(x)}$, $v \in (F_x)_{b(x)}$, $w \in (E_x)_{c(x)}$, $q \in (P_x)_{e(x)}$ satisfying the fuzzy resolvent operator equation (1.2) and so (x, u, v, w, q) is a solutions of the fuzzy nonlinear variational inclusions (1.1) and the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ and $\{q_n\}$ generated by Algorithm 2.1 converge strongly x, u, v, w and q in X, respectively.

Proof. By the λ-Lipschitz continuity of N with respect to the first argument and the μ-Lipschitz continuity with respect to the second, the ξ-Lipschitz continuity of T, the α-Lipschitz continuity of F, the ζ-Lipschitz continuity of E and the σ-Lipschitz continuity of P, we have

$$\begin{aligned}
\|N(u_n, v_n) - N(u_{n-1}, v_{n-1})\| &\le \lambda\|u_n - u_{n-1}\| + \mu\|v_n - v_{n-1}\| \\
&\le \lambda\left(1+\frac{1}{n}\right) H\big((T_{x_n})_{a(x_n)}, (T_{x_{n-1}})_{a(x_{n-1})}\big) + \mu\left(1+\frac{1}{n}\right) H\big((F_{x_n})_{b(x_n)}, (F_{x_{n-1}})_{b(x_{n-1})}\big) \\
&\le \lambda\xi\left(1+\frac{1}{n}\right)\|x_n - x_{n-1}\| + \mu\alpha\left(1+\frac{1}{n}\right)\|x_n - x_{n-1}\| \\
&\le (\lambda\xi + \mu\alpha)\left(1+\frac{1}{n}\right)\|x_n - x_{n-1}\|,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
\|w_n - w_{n-1}\| &\le \left(1+\frac{1}{n}\right) H\big((E_{x_n})_{c(x_n)}, (E_{x_{n-1}})_{c(x_{n-1})}\big) \\
&\le \zeta\left(1+\frac{1}{n}\right)\|x_n - x_{n-1}\|,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}\|q_n - q_{n-1}\| &\le \left(1+\frac{1}{n}\right) H\big((P_{x_n})_{e(x_n)}, (P_{x_{n-1}})_{e(x_{n-1})}\big)\\ &\le \sigma\left(1+\frac{1}{n}\right)\|x_n - x_{n-1}\|. \end{aligned}\tag{2.10}$$

From (2.4) by Lemma 1.1, for any $j(z_{n+1}-z_n) \in J(z_{n+1}-z_n)$ we have

$$\begin{aligned}\|z_{n+1}-z_n\|^2 &= \|g(x_n) - \eta(G(w_n) - N(u_n, v_n)) - g(x_{n-1}) + \eta(G(w_{n-1}) - N(u_{n-1}, v_{n-1}))\|^2\\ &= \|g(x_n) - g(x_{n-1}) - \eta(G(w_n) - G(w_{n-1}) - N(u_n, v_n) + N(u_{n-1}, v_{n-1}))\|^2\\ &\le \|g(x_n) - g(x_{n-1})\|^2 - 2\eta\langle G(w_n) - G(w_{n-1}) - (N(u_n, v_n) - N(u_{n-1}, v_{n-1})), j(z_{n+1}-z_n)\rangle\\ &\le \beta^2\|x_n - x_{n-1}\|^2 + 2\eta\|G(w_n) - G(w_{n-1}) - (N(u_n, v_n) - N(u_{n-1}, v_{n-1}))\|\|z_{n+1}-z_n\|. \end{aligned}\tag{2.11}$$

On the other hand, since

$$\|x+y\|^2 \le 2(\|x\|^2 + \|y\|^2) \quad \text{for } x, y \in X,$$

by (2.8), (2.9) and the ϵ-Lipschitz continuity of G,

$$\begin{aligned}&\|G(w_n) - G(w_{n-1}) + N(u_{n-1}, v_{n-1}) - N(u_n, v_n)\|^2 \le 2\|G(w_n) - G(w_{n-1})\|^2 + 2\|N(u_n, v_n) - N(u_{n-1}, v_{n-1})\|^2\\ &\le 2\epsilon^2\zeta^2\left(1+\frac{1}{n}\right)^2\|x_n - x_{n-1}\|^2 + \left[2\lambda^2\xi^2 + 4\lambda\xi\mu\alpha + 2\mu^2\alpha^2\right]\left(1+\frac{1}{n}\right)^2\|x_n - x_{n-1}\|^2\\ &= 2\left(1+\frac{1}{n}\right)^2\left(\epsilon^2\zeta^2 + \lambda^2\xi^2 + \mu^2\alpha^2 + 2\lambda\xi\mu\alpha\right)^2\|x_n - x_{n-1}\|^2. \end{aligned}\tag{2.12}$$

From (2.11) and (2.12), we get

$$\|z_{n+1}-z_n\|^2 \le \beta^2\|x_n - x_{n-1}\|^2 + 2\sqrt{2}\eta\left(1+\frac{1}{n}\right)\left(\epsilon^2\zeta^2 + \lambda^2\xi^2 + \mu^2\alpha^2 + 2\lambda\xi\mu\alpha\right)\|x_n - x_{n-1}\|\,\|z_{n+1}-z_n\|.$$

Hence

$$\|z_{n+1}-z_n\| \le \frac{b_n + \sqrt{b_n^2 + 4\beta^2}}{2}\|x_n - x_{n-1}\|,\tag{2.13}$$

where

$$b_n = 2\sqrt{2}\eta\left(1+\frac{1}{n}\right)t \quad (n \in \mathbb{N}) \text{ and } t = \epsilon^2\zeta^2 + \lambda^2\xi^2 + \mu^2\alpha^2 + 2\lambda\xi\mu\alpha.$$

On the other hand, since g is γ-strongly accretive, by Lemma 1.1, from (2.5)

for any $j(x_n - x_{n-1}) \in J(x_n - x_{n-1})$,

$$\begin{aligned}\|x_n - x_{n-1}\|^2 &= \|J_{A(\cdot,q_{n-1})}(z_n) - J_{A(\cdot,q_{n-2})}(z_{n-1}) - [g(x_n) - x_n - (g(x_{n-1}) - x_{n-1})]\|^2\\ &\le \|J_{A(\cdot,q_{n-1})}(z_n) - J_{A(\cdot,q_{n-2})}(z_{n-1})\|^2 - 2\langle g(x_n) - x_n - (g(x_{n-1}) - x_{n-1}), j(x_n - x_{n-1})\rangle\\ &= \|J_{A(\cdot,q_{n-1})}(z_n) - J_{A(\cdot,q_{n-2})}(z_{n-1})\|^2 - 2\langle g(x_n) - g(x_{n-1}), j(x_n - x_{n-1})\rangle + 2\langle x_n - x_{n-1}, j(x_n - x_{n-1})\rangle\\ &\le \|J_{A(\cdot,q_{n-1})}(z_n) - J_{A(\cdot,q_{n-2})}(z_{n-1})\|^2 + 2(1-\gamma)\|x_n - x_{n-1}\|^2. \end{aligned}\tag{2.14}$$

On the other hand, from (2.6) and (2.10) by Proposition 1.1, we obtain

$$\begin{aligned}\|J_{A(\cdot,q_{n-1})}(z_n) - J_{A(\cdot,q_{n-2})}(z_{n-1})\|^2 &\le \|J_{A(\cdot,q_{n-1})}(z_n) - J_{A(\cdot,q_{n-1})}(z_{n-1}) + J_{A(\cdot,q_{n-1})}(z_{n-1}) - J_{A(\cdot,q_{n-2})}(z_{n-1})\|^2\\ &\le 2\|J_{A(\cdot,q_{n-1})}(z_n) - J_{A(\cdot,q_{n-1})}(z_{n-1})\|^2 + 2\|J_{A(\cdot,q_{n-1})}(z_{n-1}) - J_{A(\cdot,q_{n-2})}(z_{n-1})\|^2\\ &\le 2\|z_n - z_{n-1}\|^2 + 2\kappa\|q_{n-1} - q_{n-2}\|^2\\ &\le 2\|z_n - z_{n-1}\|^2 + 2\kappa\sigma^2\left(1+\frac{1}{n-1}\right)^2\|x_{n-1} - x_{n-2}\|^2. \end{aligned}\tag{2.15}$$

From (2.14) and (2.15), we obtain

$$\|x_n - x_{n-1}\|^2 \le 2\|z_n - z_{n-1}\|^2 + 2(1-\gamma)\|x_n - x_{n-1}\|^2 + 2\kappa\sigma^2\left(1+\frac{1}{n-1}\right)^2\|x_{n-1} - x_{n-2}\|^2.$$

Thus, from (2.13)

$$\begin{aligned}
\|x_n - x_{n-1}\|^2 &\le \frac{2}{-1+2\gamma}\|z_n - z_{n-1}\|^2 + \frac{2\kappa\sigma^2\left(1+\frac{1}{n-1}\right)^2}{-1+2\gamma}\|x_{n-1}-x_{n-2}\|^2 \\
&\le \frac{2}{-1+2\gamma}\left(\frac{c_n+\sqrt{c_n^2+4\beta^2}}{2}\right)^2\|x_{n-1}-x_{n-2}\|^2 + \frac{2\kappa\sigma^2\left(1+\frac{1}{n-1}\right)^2}{-1+2\gamma}\|x_{n-1}-x_{n-2}\|^2 \\
&= \frac{1}{-1+2\gamma}\left(\frac{\left(c_n+\sqrt{c_n^2+4\beta^2}\right)^2}{2} + 2\kappa\sigma^2\left(1+\frac{1}{n-1}\right)^2\right)\|x_{n-1}-x_{n-2}\|^2
\end{aligned}$$

where $c_n = 2\sqrt{2}\eta(1+\frac{1}{n-1})^2 t \ \ (n \in \mathbb{N})$.

Finally, we have

$$\|x_n - x_{n-1}\| \le \theta_n \|x_{n-1} - x_{n-2}\|, \tag{2.16}$$

where

$$\theta_n = \left[\frac{1}{-1+2\gamma}\left(\frac{\left(c_n+\sqrt{c_n^2+4\beta^2}\right)^2}{2} + 2\kappa\sigma^2\left(1+\frac{1}{n-1}\right)^2\right)\right]^{\frac{1}{2}}.$$

Let

$$\theta = \left[\frac{1}{-1+2\gamma}\left(\frac{\left(c+\sqrt{c^2+4\beta^2}\right)^2}{2} + 2\kappa\sigma^2\right)\right]^{\frac{1}{2}},$$

where $c = 2\sqrt{2}\eta t$.

Obviously $\theta_n \to \theta$ as $n \to \infty$. It is easy to prove that condition (2.7) implies that $0 < \theta < 1$ and so $0 < \theta_n < 1$ for sufficiently large n. It follows from (2.16) that $\{x_n\}$ is a Cauchy sequence and then from (2.13) that $\{z_n\}$ is also a Cauchy sequence. Let $x_n \to x$ and $z_n \to z$. By condition (ii), it follows from (2.8)–(2.10) that $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ and $\{q_n\}$ are also Cauchy sequences. Let $u_n \to u$, $v_n \to v$, $w_n \to w$, $q_n \to q$ (as $n \to \infty$), then by (2.4) and (2.5) we have

$$\begin{aligned}
z_{n+1} &= g(x_n) - \eta(G(w_n) - N(u_n, v_n)) \\
&= J_{A(\cdot,q_{n-1})}(z_n) - \eta(G(w_n) - N(u_n, v_n)).
\end{aligned}$$

By the continuities of g and G from (2.6), letting $n \to \infty$ in the above expression to obtain

$$\begin{aligned}
z &= g(x) - \eta(G(w) - N(u, v)) \\
&= J_{A(\cdot,q)}(z) - \eta(G(w) - N(u, v)).
\end{aligned}$$

Finally, we prove that $u \in (T_x)_{a(x)}$, $v \in (F_x)_{b(x)}$, $w \in (E_x)_{c(x)}$, $q \in (P_x)_{e(x)}$. Since $u_n \in (T_{x_n})_{a(x_n)}$, we have

$$\begin{aligned}
\operatorname{dist}(u, (T_x)_{a(x)}) &\le \|u - u_n\| + \operatorname{dist}(u_n, (T_x)_{a(x)}) \\
&\le \|u - u_n\| + \operatorname{dist}(u_n, (T_{x_n})_{a(x_n)}) + H((T_{x_n})_{a(x_n)}, (T_x)_{a(x)}) \\
&\le \|u - u_n\| + 0 + \xi\|x_n - x\| \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which shows that $u \in (T_x)_{a(x)}$ due to the closedness of $(T_x)_{a(x)}$.

In similar ways, we also can prove that $v \in (F_x)_{b(x)}$, $w \in (E_x)_{c(x)}$ and $q \in (P_x)_{e(x)}$, which imply that (x, z, u, v, w, q) is a solution of fuzzy resolvent operator equation problem (1.2). Hence by Theorem 2.1 (x, u, v, w, q) is a solution of fuzzy nonlinear variational inclusion problem (1.1), and the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ and $\{q_n\}$ generated by Algorithm 2.1, converge strongly x, u, v, w and q in X, respectively. □

Remark 2.1. Our result improves and generalizes many results in [1,5–7,12].

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